Seismic Wave Modelling

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Discretization

We wish to numerically model the propagation of seismic waves through a certain volume of space over a period of time. The first step will be to discretize that space and time.

We will represent the continuous physical space as a grid of discrete cells. We can then represent physical quantities within that space using numbers associated with each cell.

We will repeatedly update the numbers in each grid cell, each update constituting a discrete step in time. Thus the changing numbers will model the wave propagation through that space over time.

The Wave Equation

Seismic waves are waves of energy that travel through the earth. Since the earth is mostly solid, and most solid materials are elastic, these waves are best described by the elastic wave equation.

However, the elastic wave equation is computationally expensive. As a cheaper approximation, we use the acoustic wave equation:

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p$$

Where p is the acoustic pressure, c is the speed of sound, t is time and ∇^2 is the Laplace operator.

The Laplace Operator

Given a function f at a point p, i.e. $f(p)$, the Laplacian $\nabla^2 f(p)$ is the rate at which the average value of f over spheres centred at p deviates from $f(p)$ as the spheres grow.

In a Cartesian co-ordinate system, the Laplacian is given by the sum of the second partial derivatives of f with respect to each of the spatial variables. For example, in 2D it would be this:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

And in 3D it would be this:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Finite Difference Method

The wave equation is a partial differential equation. There are various numerical methods that can be used to compute approximations to the solutions of such equations.

We will use the finite difference method. I have written another set of slides describing this method, which can be found here:

`http://www.collicutt.co.uk/finitedifference.pdf`

Discretizing Time

Using the finite difference technique, we can calculate an approximation to $f''(x)$ like so:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Let our *step-size* in time be dt, then plug this approximation into the acoustic wave equation:

$$\frac{p_{t+1} - 2p_t + p_{t-1}}{dt^2} \approx c^2 \nabla^2 p_t$$

Then rearrange to get a formula for calculating the next state of p from the current and previous states:

$$p_{t+1} \approx 2p_t - p_{t-1} + dt^2 c^2 \nabla^2 p_t$$

Discretizing Space

We will use similar approximations to compute the spatial derivatives used to calculate the Laplacian.

For example, to compute the second-order derivative in x we could use the same approximation as we used on the previous slide:

$$\frac{\partial^2 p}{\partial x^2} \approx \frac{p_{x+1} - 2p_x + p_{x-1}}{dx^2}$$

Or we could compute a more accurate result by using a higher-order approximation:

$$\frac{\partial^2 p}{\partial x^2} \approx \frac{-\frac{1}{12}p_{x+2} + \frac{4}{3}p_{x+1} - \frac{5}{2}p_x + \frac{4}{3}p_{x-1} - \frac{1}{12}p_{x-2}}{dx^2}$$

Anisotropy

In our model so far, the speed of wave propagation is the same in all directions. The wave velocity is determined by a single scalar term in our equation: c.

In reality, different materials in the earth will propagate waves at different speeds in different directions. For example, waves will propagate faster along layers of sediment than perpendicularly through those layers.